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Lie algebras and v_n-periodic spaces.

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Goal: generalize Quillen's $\mathbb{Q}$-homotopy theory to telescopic localizations.

Quillen. $(S_*^{\geq 2})_{\mathbb{Q}} \xrightarrow{C_*(X,\mathbb{Q})} \mathrm{coCAlg}(\mathrm{Ch}_{\mathbb{Q}})^{\geq 2}$

Lower central series of simplicial group is a dg Lie alg. (equivalence $\|$ down to $\mathrm{Lie}(\mathrm{Ch}_{\mathbb{Q}})^{\geq 1}$)

TAQ / Chevalley-Eilenberg (between $\mathrm{Lie}(\mathrm{Ch}_{\mathbb{Q}})^{\geq 1}$ and $\mathrm{coCAlg}(\mathrm{Ch}_{\mathbb{Q}})^{\geq 2}$)

$\mathrm{Lie}(\mathrm{Ch}_{\mathbb{Q}})^{\geq 1}$: connected dg Lie algebras over $\mathbb{Q}$

$\mathrm{coCAlg}(\mathrm{Ch}_{\mathbb{Q}})^{\geq 2}$: dg cocommutative coalgebras over $\mathbb{Q}$.

Fix p.

Rational homotopy theory is the first of a seq. of homology theories,

$H\mathbb{Q}, K(1), K(2), K(3), \ldots$

$K(1)$ "=" KU/p

$K(2), K(3), \ldots$: Morava K-theories.

Def. A finite pointed space V is of type n,

$$\tilde{K}(i)_* V = 0,$$

$i<n$. $\tilde{K}(n)_* V \neq 0$.

Def. A v_n-self map is a map $\Sigma^d V \xrightarrow{v} V$ s.t.

$$\tilde{K}(i)_* v = \begin{cases} \text{nilpotent} & i \neq n, \\ \text{iso} & i = n. \end{cases}$$

Ex. S^k is a type 0 space.

$S^k \xrightarrow{p} S^k$ is a v_0-self map, $k \geq 1$.

Ex. S^k/p is a type 1 space.

$KU/p(S^k/p)$ is non-zero. Adams map

$$\alpha: \Sigma^d S^k/p \longrightarrow S^k/p$$

p odd,
$d = 2p-1$, $k \geq 3$.

α induces an iso in K-theory. It is a v_1-map.
Induces some power of β.

Ex. $\mathrm{cof}(\alpha)$ is a type 2 space. Non-obvious: has a v_2-self map.

Thm (Mitchell). Type n spaces exist for all n.

Thm (Hopkins–Smith). A finite type n space admits a v_n-self map (after sufficiently many suspensions).

v_n-periodic homotopy groups.

V type n,

v v_n-self map, $\Sigma^d V \to V$.

X a pointed space.

$$\pi_* \operatorname{Map}_*(V, X) \circlearrowleft v$$

Invert v. Get v-periodic homotopy groups of X.

Ex. $V = S^n$, $v = p$, get rational homotopy groups.

More precisely: telescopic functor

$$\Phi_v : S_* \longrightarrow Sp$$

associated to v,

$$\Phi_v(X)_0 = \operatorname{Map}_*(V, X),$$

$$\Phi_v(X)_d = \operatorname{Map}_*(V, X),$$

$$\vdots$$

$$\Phi_v(X)_{kd} = \operatorname{Map}_*(V, X).$$

$$\begin{array}{ccc} \Phi_v(X)_{kd} & \longrightarrow & \Omega^d \Phi_v(X)_{(k+1)d} \\ \wr\| & & \wr\| \\ \operatorname{Map}_*(V,X) & \xrightarrow{v^*} & \operatorname{Map}_*(\Sigma^d V, X). \end{array}$$

Observe: $\pi_* \Phi_v(X)$ are exactly the v-periodic homotopy groups of X.

Depends on V, but not on v.

The various Φ_V can be packaged together into the Bousfield - Kuhn functor.

$$\Phi : S_* \longrightarrow Sp_{T(n)}$$

$\uparrow$ T(n)-local spectra.

Of a v_n-self map on a finite type n spectra.

Properties. ($n \geq 1$).

1) $\Phi_V(X) \simeq \mathbb{D}V \wedge \Phi(X)$ "natural in V".

$\uparrow$ SW dual

2) $\Phi\Omega^\infty : Sp \longrightarrow Sp_{T(n)}$ is equivalent to T(n)-localization, $L_{T(n)}$.

So, $L_{T(n)}$ depends only on the connective cover.

Also, doesn't need the infinite loop space structure.

Rem. Φ is essentially char. by 1).

Def. A map f of pointed spaces is a v_n-periodic $\simeq$ if $\Phi(f)$ is an $\simeq$.

Thm (Bousfield – Dror-Farjoun). a) The localization of S_* at the v_n-periodic equivalences exists. More precisely, there exists a functor $\Pi : S_* \to S_*^{v_n}$ s.t.

$$\mathrm{Fun}(S_*^{v_n}, \mathcal{C}) \xrightarrow[\text{f.f.}]{\Pi^*} \mathrm{Fun}(S_*, \mathcal{C})$$

with ess. image those functors

$$S_* \to \mathcal{C}$$

inverting the v_n-equivalences.

But, Π is not Bousfield. It does not preserve localizations.

b) $\Phi : S_*^{v_n} \to Sp_{T(n)}$ admits a left adjoint Θ.

Lie alg. free forget   Coalgebra cofree, forget.

Rem.

$$\boxed{Sp_{T(n)} \underset{\Phi}{\overset{\Theta}{\rightleftarrows}} S_*^{v_n}} \underset{\Omega^\infty_{T(n)}}{\overset{\Sigma^\infty_{T(n)}}{\rightleftarrows}} Sp_{T(n)}.$$

Since $\Phi \circ \Omega^\infty_{T(n)} \simeq \mathrm{id}_{Sp_{T(n)}}$, $\quad \Sigma^\infty_{T(n)} \circ \Theta \simeq \mathrm{id}_{Sp_{T(n)}}$.

Maybe v_n box. Only at n.
Not integral.

Thm.$^{(*)}$ There is an equivalence of ∞-cats

$$S_*^{v_n} \simeq \mathrm{Lie}(Sp_{T(n)}).$$

↑ Ching's operad using Goodwillie derivatives.

The composition

$$S_*^{v_n} \simeq \mathrm{Lie}(Sp_{T(n)}) \xrightarrow{\text{forget}} Sp_{T(n)}$$

is $\simeq$ to Φ.

Thm. There is a functor

$$S_*^{v_n} \xrightarrow{C_{T(n)}} \mathrm{coCAlg}(Sp_{T(n)}).$$

↑ $\Sigma^\infty_{T(n)}$ + coalgebra structure.

It is fully faithful on spaces which are Φ-good, i.e., those X for which

$$\Phi X \xrightarrow{\sim} \lim_k \Phi P_k \mathrm{id}(X).$$

↑ Should be δ_k.

Ex. Spheres are Φ-good (Arone–Mahowald). Moore spaces S^k/p are not.

Outline of proof of (*). → Joint with Eldred – Mathew – Meier.

Part A. The adjoint pair is monadic, i.e.,

$$S_*^{v_n} \simeq \mathrm{Alg}_{\Phi\Theta}(Sp_{T(n)}).$$

$$\cdots \rightrightarrows (\Phi\Theta)^2 E \rightrightarrows (\Phi\Theta) E \longrightarrow E.$$

By Barr–Beck–Lurie, suffices to check that (1) Φ is conservative (← Basically by construction.) and (2) that Φ preserves geometric realizations. Part (2) boils down to the fact that

$$\mathrm{Map}_*(V, -)$$

preserves such colimits for highly connected spaces.
Use that Ω preserves geo. realizations of connected spaces.

Part B. Understand $\Phi\Theta$.

Sub-Thm. $(\Phi\Theta)(E) \simeq L_{T(n)} \bigvee_{k \geq 1} (\delta_k \, \mathrm{id} \wedge E^{\wedge k})_{h\Sigma_k}$.

($\delta_k \, \mathrm{id}$ ← SW dual of partition complex.)

proof. Kuhn.

$$\Sigma^\infty_{T(n)} \Omega^\infty_{T(n)} E \simeq L_{T(n)} \bigvee_{k \geq 1} (E^{\wedge k})_{h\Sigma_k}.$$

Consequence. Any functor $Sp_{T(n)} \xrightarrow{F} Sp_{T(n)}$ which preserves sifted colimits is of the form

$$F(X) \simeq L_{T(n)} \bigvee_{k \geq 1} (C(k) \wedge X^{\wedge k})_{h\Sigma_k} \text{ for some } k.$$

($C(k)$ ← $\delta_k F$)

Back to B, the claim follows immediately.
Now, have to check that actually get an
$\simeq$ of functors from $\Phi \Theta$ and the Lie monad.

Now, $\{S_k\ \mathrm{id}\}$ is the operad Koszul dual to $\mathrm{Com}^{\vee E_\infty}$.

here: $\mathrm{id}_{S_+^{vn}} \to \Omega^\infty_{T(n)} \Sigma^\infty_{T(n)} \rightrightarrows \cdots$

$$\Phi\Theta \to \Phi(\Omega^\infty_{T(n)} \Sigma^\infty_{T(n)})\Theta \rightrightarrows \Phi(\Omega^\infty_{T(n)} \Sigma^\infty_{T(n)})^2\Theta \rightrightarrows$$

$$\wr| \qquad\qquad \wr|$$

$$\mathrm{id}_{Sp_{T(n)}} \rightrightarrows \Sigma^\infty_{T(n)} \Omega^\infty_{T(n)} \rightrightarrows \cdots$$

Gives a map

$$\Phi\Theta \to \mathrm{Tot}\left((\Sigma^\infty_{T(n)}, \Omega^\infty_{T(n)})^\bullet \right)$$

$$\wr)$$

$$\mathrm{Tot}\left(\left(\mathrm{Sym}^{\geq 1}_{T(n)}\right)^\bullet \right)$$

$$\wr|$$

Monad map. $\to$ cobar construction of comm. cooperad.

Actually an $\simeq$ by work above, sub-thm.

That finishes the proof: $\Phi\Theta \simeq \mathrm{Lie}$.